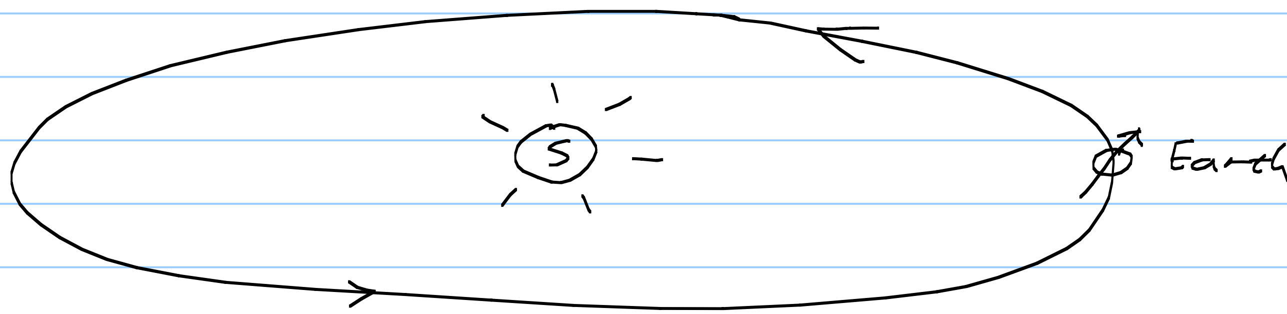
# Spin

Note Title

Earth – Sun analogy




* Orbital angular momentum of earth
  $\vec{L}$ : rotation of earth around sun

* Spin angular momentum of earth
  $\vec{S}$ : rotation of earth around the south – north pole

Classicaly $\begin{cases} \vec{L} = \vec{r} \times \vec{p} \\ \vec{S} = I \vec{\omega} \end{cases}$ , but

$\vec{S}$ is nothing more than the sum of the orbital angular momenta of the component materials (for earth, rocks and soils, etc.)

* In quantum mechanis, especially for elementary particles, such as electron, spin is an intrinsic property of

the particle, and we cannot define its wave function such as $Y_\ell^m(\theta, \varphi)$.

Using the analogy to the orbital angular momentum $\vec{L}$, we start from the same commutatation relations, i.e.

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x$$
$$[S_z, S_x] = i\hbar S_y.$$

In the previous lectures, we have seen that through algebraic methods, these commutation relations automatically generate the full spectrum of eigenvalues for $S^2$ and $S_z$ such that

$$S^2 |s\, m\rangle = \hbar^2 s(s+1) |s\, m\rangle$$
$$S_z |s\, m\rangle = \hbar m |s\, m\rangle$$

For comparison, the equivalent equations for the orbital angular momentum were

$$L^2 Y_\ell^m(\theta, \varphi) = \hbar^2 \ell(\ell+1) Y_\ell^m(\theta, \varphi)$$
$$L_z Y_\ell^m(\theta, \varphi) = \hbar m Y_\ell^m(\theta, \varphi)$$

Note that for spin, it is impossible to define the eigenstate as a spatial coordinate

such as $(\theta, \varphi)$.
So the Dirac notation is almost a must for the spin case.
If we use the Dirac notation even for the orbital angular momentum,

$$Y_\ell^m(\theta, \varphi) = |\ell\, m\rangle$$

But

$$|s, m\rangle \neq Y_s^m(\theta, \varphi)$$

The spin eigenstate is **NOT** spherical harmonics or any other function of $(\theta, \varphi)$

* The ladder operators are also defined in the same way as before

$$S_\pm |s\, m\rangle = A_s^m |s\, (m \pm 1)\rangle$$

, where $S_\pm = S_x \pm i S_y$ and $A_s^m$ is just some constant for normalization, and gives as $A_s^m = \hbar\sqrt{s(s+1) - m(m \pm 1)}$ proven in G. prob. 4.18.

* $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \quad 1$
  and $m = -s, -s+1, \ldots s-1, s$

Every elementary particle has its own permanent $s$ value

Ex.)

| $s$ | name of elementary particle |
|---|---|
| $1/2$ | electron, quarks, ... |
| $1$ | photon |
| $3/2$ | delta |
| $\vdots$ | $\vdots$ |

But "$\ell$" can take any values.
(e.g. orbital angular momentum of the electron in a hydrogen atom).

$\boxed{\text{Spin } 1/2}$ $\quad s = \frac{1}{2}$

$$|s\ m\rangle \begin{cases} |\frac{1}{2}\ \frac{1}{2}\rangle \equiv |\uparrow\rangle & \text{spin up} \\ |\frac{1}{2}\ -\frac{1}{2}\rangle \equiv |\downarrow\rangle & \text{spin down} \end{cases}$$

Any state in the Hilbert space spanned by these two basis states can be written as

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

with $\chi_+ \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\uparrow\rangle = |1\rangle$

and $\chi_- \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle = |2\rangle$

$\chi$ is called a "spinor"

In this spin 1/2 space, all operators can be expressed as 2 × 2 matrices.

$$S^2 \chi_\pm = S^2 |\tfrac{1}{2} \pm\tfrac{1}{2}\rangle = \hbar^2 \tfrac{1}{2}\cdot\tfrac{3}{2} \cdot |\tfrac{1}{2} \pm \tfrac{1}{2}\rangle$$
$$= \tfrac{3}{4}\hbar^2 \chi_\pm$$

$$\Rightarrow \quad \langle 1|S^2|1\rangle = \langle 2|S^2|2\rangle = \tfrac{3}{4}\hbar^2$$
$$\langle 1|S^2|2\rangle = \langle 2|S^2|1\rangle = 0$$

$$\text{So} \quad S^2 = \begin{pmatrix} \langle 1|S^2|1\rangle & \langle 1|S^2|2\rangle \\ \langle 2|S^2|1\rangle & \langle 2|S^2|2\rangle \end{pmatrix}$$
$$= \tfrac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_z \chi_+ \ (= S_z|1\rangle) = \tfrac{\hbar}{2} \chi_+$$

$$S_z \chi_- \ (= S_z|2\rangle) = -\tfrac{\hbar}{2} \chi_-$$

$$\Rightarrow \quad \langle 1|S_z|1\rangle = \tfrac{\hbar}{2}, \quad \langle 2|S_z|2\rangle = -\tfrac{\hbar}{2}$$
$$\langle 1|S_z|2\rangle = \langle 2|S_z|1\rangle = 0$$

$$\Rightarrow \quad S_z = \tfrac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Also from $S_\pm |s\, m\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\ |s\,(m\pm1)\rangle$

$$S_+ |\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle = \hbar\sqrt{\tfrac{3}{4} - (-\tfrac{1}{2})\tfrac{1}{2}}\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$S_- |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{3}{4} - \tfrac{1}{2}(-\tfrac{1}{2})}\ |\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle = \hbar|\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$$

$S_+|\uparrow\rangle = 0$

So $S_+|2\rangle = \hbar|1\rangle$, $S_+|1\rangle = 0$

$S_-|1\rangle = \hbar|2\rangle$, $S_-|2\rangle = 0$

$S_-|\downarrow\rangle = 0$

$$\Rightarrow \langle 1|S_+|1\rangle = \langle 2|S_+|2\rangle = 0$$
$$\langle 1|S_+|2\rangle = \hbar$$
$$\langle 2|S_+|1\rangle = 0$$

$$\Rightarrow S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Similarly $S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

From $S_\pm = S \pm i S_y$

$$\Rightarrow S_x = \frac{1}{2}(S_+ + S_-)$$

and $S_y = \frac{1}{2i}(S_+ - S_-)$

$$\therefore S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

So if we define $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are called Pauli matrices.

$S_x$, $S_y$, $S_z$, and $S^2$ are hermitian
But $S_\pm$ are not hermitian.

* With $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ a general spinor,
What are the probabilities of measuring $S_z$ of $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$?

$$\chi = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\underbrace{|\uparrow\rangle}_{S_z = \frac{\hbar}{2}} + b\underbrace{|\downarrow\rangle}_{S_z = -\frac{\hbar}{2}}$$

So $P(\frac{\hbar}{2}) = |a|^2$ and $P(-\frac{\hbar}{2}) = |b|^2$
with $\chi$ normalized properly.

$\boxed{\text{Ex.}}$ For $\chi = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$
What are possible values for $S_x$ measurement and their probabilities?

$\boxed{\text{Answer:}}$ $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let's find its eigen values and eigen states.

$$S_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$$

$$\Rightarrow \lambda = \pm\frac{\hbar}{2}$$

For $\lambda = \frac{\hbar}{2}$

$$-\frac{\hbar}{2}\alpha + \frac{\hbar}{2}\beta = 0 \Rightarrow \beta = \alpha$$

$$\Rightarrow \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

For $\lambda = -\frac{\hbar}{2}$

$$\Rightarrow \frac{\hbar}{2}\alpha + \frac{\hbar}{2}\beta = 0 \Rightarrow \beta = -\alpha$$

$$\Rightarrow \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

So if $\chi = a|\chi_+^{(x)}\rangle + b|\chi_-^{(x)}\rangle$

$$\Rightarrow P(S_x = \frac{\hbar}{2}) = |\langle\chi_+^{(x)}|\chi\rangle|^2$$

$$= |a|^2$$

$$P(S_x = -\frac{\hbar}{2}) = |\langle\chi_-^{(x)}|\chi\rangle|^2$$

$$= |b|^2$$

Thus $P(S_x = \frac{\hbar}{2}) = \left|\frac{1}{\sqrt{2}}(1,\ 1)\frac{1}{\sqrt{6}}\begin{pmatrix}1+i\\ 2\end{pmatrix}\right|^2$

$$= \left|\frac{1}{\sqrt{12}}(1+i+2)\right|^2$$

$$= \frac{1}{12}(9+1) = \frac{5}{6}$$

$$P(S_x = -\frac{\hbar}{2}) = \left| \frac{1}{\sqrt{2}}(1, -1)\frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{12}}(1+i-2) \right|^2$$

$$= \frac{1}{12}(1+1) = \underline{\frac{1}{6}}$$

What is $\langle S_x \rangle$ here?

Two methods

① $$\langle S_x \rangle = P\left(\frac{\hbar}{2}\right) \cdot \frac{\hbar}{2} + P\left(-\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)$$

$$= \frac{5}{6}\frac{\hbar}{2} + \frac{1}{6}\left(-\frac{\hbar}{2}\right)$$

$$= \frac{4\hbar}{12} = \frac{\hbar}{3}$$

② $$\langle S_x \rangle = \langle \chi | S_x | \chi \rangle$$

$$= \frac{1}{\sqrt{6}}(1-i, 2) \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

$$= \frac{\hbar}{12}(1-i, 2)\begin{pmatrix} 2 \\ 1+i \end{pmatrix} = \frac{\hbar}{12}(2-2i+2+2i)$$

$$= \frac{4\hbar}{12} = \frac{\hbar}{3}$$

* If a spin 1/2 particle is in $\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$ state, what are the possible values of the z-component of the spin angular momentum?

$\Rightarrow \frac{\hbar}{2}$, 100 %

* Now if you measure x-component of the spin angular momentum, what are the possible values and their probabilities?

Since $\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $\chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$,

$$P\left(S_x = \frac{\hbar}{2}\right) = \left|\langle \chi_+^{(x)} | \chi_+ \rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}(1, 1)\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$

$$P\left(S_x = -\frac{\hbar}{2}\right) = \left|\langle \chi_-^{(x)} | \chi_+ \rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}(1, -1)\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$

* Now if your measurement of $S_x$ yielded $\frac{\hbar}{2}$, what is the state of the particle after the measurement?

$\Rightarrow \chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$